

# A Comparison of Turbulent Thermal Convection Between Conditions of Constant Temperature and Constant Flux

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We report the results of high resolution direct numerical simulations of two-dimensional Rayleigh-Bénard convection for Rayleigh numbers up to  $Ra = 10^{10}$  in order to study the influence of temperature boundary conditions on turbulent heat transport. Specifically, we considered the extreme cases of fixed heat flux (where the top and bottom boundaries are poor thermal conductors) and fixed temperature (perfectly conducting boundaries). Both cases display identical heat transport at high Rayleigh numbers fitting a power law  $Nu \approx 0.138 \times Ra^{.285}$  with a scaling exponent indistinguishable from  $2/7 = .2857\dots$  above  $Ra = 10^7$ . The overall flow dynamics for both scenarios, in particular the time averaged temperature profiles, are also indistinguishable at the highest Rayleigh numbers. The findings are compared and contrasted with results of recent three-dimensional simulations.

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## I. INTRODUCTION

Convection refers to phenomena where spatial inhomogeneities in an advected scalar field drive a fluid flow which in turn transports the scalar. Convection of various sorts (thermal, compositional, double-diffusive) plays a fundamental role in a wide range of geophysical, astrophysical and engineering applications. Transport properties of convective flows are of utmost interest and are the focus of scientific efforts worldwide. Rayleigh-Bénard convection, the buoyancy driven flow in a fluid layer heated from below, is one of the fundamental paradigms of nonlinear physics, complex dynamics and pattern formation [1]. Despite the great deal of effort that has been devoted to it, however, the bulk transport properties of turbulent Rayleigh-Bénard convection still present challenges for theory, simulation and experiment.

The Rayleigh number  $Ra$  is a ratio of driving due to buoyancy resulting from a vertical temperature gradient, to damping due to the fluid's viscosity and thermal diffusion. The enhancement of vertical heat transport by the convectively driven flow is measured by the dimensionless Nusselt number  $Nu$ . The Prandtl number  $Pr = \nu/\kappa$ , is the ratio of the fluid's kinematic viscosity to its thermal diffusivity. The goal of many experiments, simulations, theories and analyses is to discern the behavior of  $Nu$  as a function of  $Ra$  and  $Pr$  and geometric structure (often aspect ratio) of the domain. When the Prandtl number and aspect ratio are fixed, the Rayleigh number enjoys the status of the control parameter. Bulk transport in convective turbulence is an open problem: there is still no universally accepted theoretical expectation for what the asymptotic high Rayleigh number  $Nu$ - $Ra$  relationship should be [2, 3, 4, 5, 6, 7], and the state of affairs experimentally is unresolved and even somewhat controversial

[8, 9, 10, 11, 12, 13, 14, 15].

Analysis has played and continues to play an important role in this problem. Mathematically reliable limits on convective transport derived from the fundamental model place constraints on theories invoking uncontrolled approximations or incorporating additional assumptions. The classic rigorous result for Rayleigh-Bénard convection between fixed-temperature no-slip plates is the scaling bound  $Nu \leq cRa^{1/2}$  where the prefactor  $c$  is uniform in  $Pr$  for fixed temperature and no-slip velocity boundary conditions [16, 17]. The  $Nu \sim Ra^{1/2}$  scaling has been proposed in several theories [2, 3, 7], albeit with distinct Prandtl number dependences, while experimental results have suggested that the asymptotic high- $Ra$  exponent is somewhere between  $2/7$  and  $1/2$  depending on the Prandtl number and, perhaps, other features. The  $1/2$  scaling bound remains the best known rigorous estimate for the arbitrary- $Pr$  problem, although  $Nu \lesssim (Ra \times \ln Ra)^{1/3}$  holds for sufficiently high  $Pr$  at fixed  $Ra$  [18] and for infinite  $Pr$  uniformly in  $Ra$  [19].

One proposed explanation for discrepancies among experimental results is the effect of finite conductivity of the upper and lower boundaries [20, 21]. Fixed-temperature (Dirichlet) boundary conditions model plates of infinite thermal conductivity while the limit of poorly conducting boundaries corresponds to fixed heat flux (Neumann) boundary conditions. Intermediate situations are modeled by interpolating (radiation) boundary conditions. It is well known that the thermal boundary conditions have a significant effect near the convective transition, decreasing the critical Rayleigh number for the onset of convection and shifting the instability to larger scales [22]. Indeed, the long-wave nature of the fixed-flux linear instability has been proposed as an important effect for pattern selection in high- $Ra$  turbulent convection in large

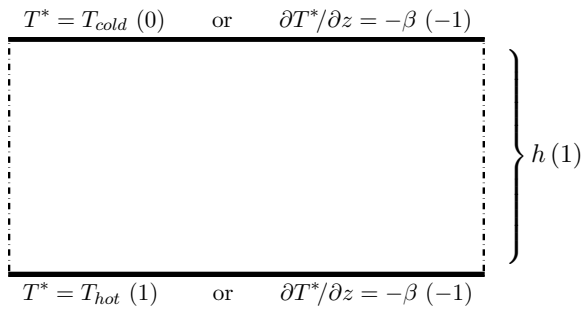


FIG. 1: 2D convection cell, periodic in the horizontal direction, indicating the different temperature boundary conditions considered: fixed temperature and fixed heat flux (non-dimensional values appear in parentheses).

aspect ratio systems [23].

Recent direct numerical simulations in a three-dimensional cylindrical cell ( $\text{Pr} = .7$ ) indicated that  $\text{Nu}$  is suppressed for finite conductivity plates above  $\text{Ra} = 10^9$ , and it was suggested that typical experimental conditions with finite conductivity plates are closer to the fixed flux, rather than the fixed temperature case [24]. Rigorous heat transport bounds for the fixed flux case were previously derived [25], but they scale  $\sim \text{Ra}^{1/2}$  as in the fixed temperature case, well above the simulation results for either boundary conditions.

In this letter we report the results of a high-resolution computational study of the difference between fixed flux and fixed temperature Rayleigh-Bénard convection in two spatial dimensions. We restrict attention to two dimensions in order to guarantee full resolution of the boundary layers at high Rayleigh numbers [26]. Not unexpectedly, the critical Rayleigh number at convective onset and  $\text{Nu}$  immediately above onset differ, with the fixed flux Nusselt number exceeding the fixed temperature Nusselt number as anticipated by linear stability analysis [22], but the Nusselt numbers and other qualitative and quantitative features of the flows are observed to coincide to a very high degree of accuracy at high Rayleigh numbers in distinction from the computational results reported for three dimensions.

## II. MATHEMATICAL MODEL AND NUMERICAL SCHEME

The flow is modeled by the Boussinesq approximation for a unit density incompressible fluid in a horizontally periodic two-dimensional domain of height  $h$  with no-slip velocity boundary conditions at the top and bottom plates (Fig. 1). After a change of variables depending on the temperature boundary conditions, the governing

equations are

$$T_t + (\mathbf{u} \cdot \nabla)T = (\text{Pr}R)^{-1/2}\Delta T, \quad (1)$$

$$\partial_t \omega + (\mathbf{u} \cdot \nabla)\omega = (\text{Pr}/R)^{1/2}\Delta \omega - \partial T / \partial x, \quad (2)$$

$$\Delta \psi = \omega, \quad \psi|_{z=0,1} = 0, \quad \partial \psi / \partial z|_{z=0,1} = 0, \quad (3)$$

where  $T$  the temperature,  $\mathbf{u} = \mathbf{i}u + \mathbf{k}w = \mathbf{i}\partial_z \psi - \mathbf{k}\partial_x \psi$  is the velocity field, and  $\omega = \partial_z u - \partial_x w$  is the vorticity. The control parameter  $R$  in the measure of the imposed thermal forcing whose definition depends on the boundary conditions:

For the case of fixed temperature at each boundary the space, time, and temperature scales are, respectively,  $h$ ,  $(h/\alpha g \delta T^*)^{1/2}$ , and  $\delta T^* = T_{\text{hot}} - T_{\text{cold}}$ , the dimensional temperature drop, where  $g$  is the acceleration due to gravity and  $\alpha$  is the thermal expansion coefficient of the fluid. Then the thermal forcing parameter  $R$  in (1) and (2) is the usual Rayleigh number

$$R = \text{Ra} = \alpha g \delta T^* h^3 / \nu \kappa \quad (4)$$

with temperature boundary conditions  $T|_{z=0} = 1$  and  $T|_{z=1} = 0$ . For the case of an imposed vertical fixed heat flux  $\sim \beta$ , the space, time, and temperature scales are, respectively,  $h$ ,  $(\alpha g \beta)^{-1/2}$ , and  $\beta h$ . Then in place of (4),

$$R = \hat{R} = \alpha g \beta h^4 / \nu \kappa \quad (5)$$

with  $\partial T / \partial z|_{z=0,1} = -1$ .

Denote the space-time average of a function  $f(\mathbf{x}, t)$  by  $\langle f \rangle$ . Then for the fixed-temperature boundary conditions the Nusselt number is

$$\text{Nu} = 1 + (\text{Pr}R)^{1/2} \langle wT \rangle, \quad (6)$$

while for the fixed-flux case it is

$$\text{Nu} = \left( 1 - (\text{Pr}\hat{R})^{1/2} \langle wT \rangle \right)^{-1}. \quad (7)$$

Note that  $\hat{R} = \text{RaNu}$  [24, 25].

The numerical scheme used for simulating (1)-(3) is a Fourier-Chebyshev spectral collocation method in space with classical fourth order Runge-Kutta for the time stepping [27]. Computation of the momentum equation (2) and the kinematic constraint (3) are decoupled through use of a high order local formula for the vorticity at the boundary, derived from the Neumann boundary condition  $\partial \psi / \partial z|_{z=0,1} = 0$  for the stream function. The Dirichlet boundary condition  $\psi|_{z=0,1} = 0$  is imposed in the solution of the elliptic system (3), which is solved by the matrix-diagonalization procedure [28, 29]. To ensure benchmark quality simulations, the grid sizes were chosen with a minimum of eight grid points in the thermal boundary layer, defined as the distance from the boundary at which the extrapolation of the linear portion of the mean profile at the boundary equals the central mean temperature [24, 30]. Achieving this resolution over the

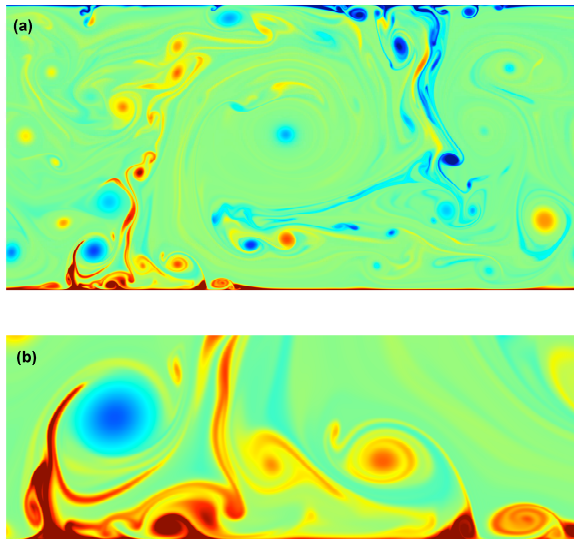


FIG. 2: Temperature snapshot for fixed flux convection at  $Ra = 1.05 \times 10^{10}$  ( $\hat{R} = 1.07 \times 10^{12}$ ): (a) full domain, (b) zoom image of boundary layer to illustrate resolution.

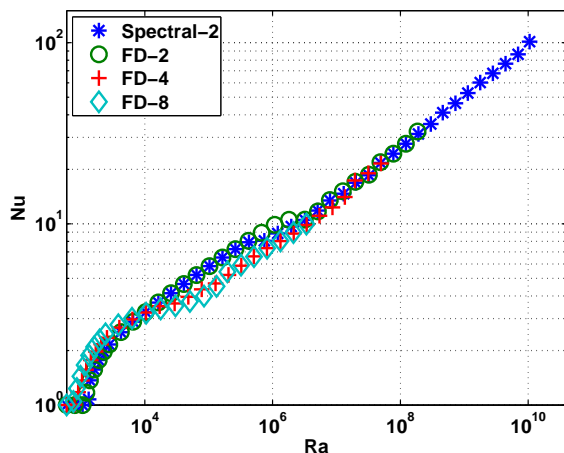


FIG. 3: Ra-Nu data for fixed-flux simulations. The spectral code was used for an aspect ratio 2 cell (Spectral-2), and the fourth order finite difference method [32] was used for cells of aspect ratio 2 (FD-2), 4 (FD-4), and 8 (FD-8).

full range of  $Ra$ , while maintaining computational efficiency and accuracy, was made possible through the use of the Kosloff & Tal-Ezer mapping [31] applied to the Chebyshev points and a high order exponential filter to control aliasing errors.

### III. RESULTS AND DISCUSSION

All results reported here are for  $Pr = 1$ . Simulations were first performed for the fixed heat flux case in a cell of aspect ratio 2. Beginning at  $\hat{R} = 600$ —below onset of

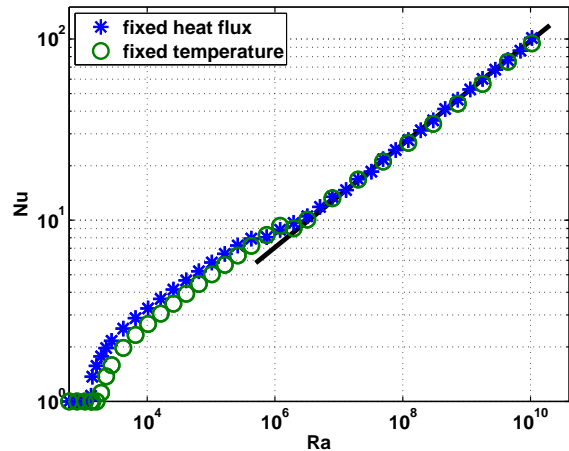


FIG. 4: Comparison of heat transport data for fixed-flux and fixed-temperature simulations in a cell of aspect ratio 2.

convection— $\hat{R}$  was increased step-by-step by a constant factor, allowing at the flow to settle into a steady or statistically stationary dynamical state before proceeding. A snapshot of the temperature field at the highest Rayleigh number is shown in Fig. 2. For each value of  $\hat{R}$  the Nusselt number  $Nu$  was measured and the effective Rayleigh number recovered from  $Ra = \hat{R}/Nu$ .

In order to validate the fixed flux simulations before moving to the fixed temperature case, the numerical experiment was repeated using a fourth order finite difference method [32] following the same protocol for cells of aspect ratio 2, 4, and 8, over a more restricted range of  $Ra$  in order to achieve the same numerical accuracy as in the spectral simulations. Fig. 3 shows all these fixed-flux  $Ra$ - $Nu$  data sets, noting that the simulation times were sufficiently long to ensure that uncertainties in the  $Nu$  measurements are within the size of plot symbols. For fixed flux Rayleigh-Bénard convection the critical value of  $\hat{R}$  at onset depends on (decreases with) the aspect ratio, and the wavelength at onset is set by the horizontal—rather than the vertical—scale of the domain [22]. Hence the initial pair of convection rolls is as wide as the aspect ratio permitted. But at each aspect ratio, as the Rayleigh number was increased the rolls eventually became unstable, turbulent convection sets in, and the flow was observed to organize itself into pairs of turbulent aspect-ratio 2 cells like those in Fig. 2. These observations, together with the data in Fig. 3, suggest that two-dimensional fixed-flux convective turbulence is independent of the aspect ratio at asymptotically high Rayleigh numbers.

Following the same protocol, simulations with fixed temperatures at the boundaries were carried out for the same values of  $Ra$  in an aspect ratio 2 cell. Fig. 4 shows the  $Ra$ - $Nu$  data for both scenarios. It is observed that the data are the same, within the simulations' uncertainties, at high Rayleigh numbers. A fit of the eight highest

Ra data points generated with the fixed flux boundary conditions yields  $Nu \approx 0.138 \times Ra^{.285}$  with a scaling exponent indistinguishable from  $2/7 = .2857\dots$ . This Nu-Ra relationship is in remarkably close agreement with the latest high-precision three-dimensional simulations for fixed-temperature conditions in a cylinder with insulating walls [33] which produced  $Nu = 0.175 \times Ra^{.283}$  for  $10^7 \leq Ra \leq 10^9$  at  $Pr = .7$ .

The quantitative correspondence between fixed-flux and fixed-temperature (two-dimensional) turbulent convection is not limited to the bulk heat flux. The mean temperature profiles are observed to converge at high Rayleigh numbers as well. Fig. 5 shows horizontally and temporally averaged temperature profiles for the two scenarios. At the highest Rayleigh numbers they are effectively indistinguishable. We have not systematically compared other statistical quantities, but we have noted striking similarities between the large scale dynamics at fixed flux and fixed temperature [34].

The  $Nu \sim Ra^{2/7}$  relationship was previously observed with fixed temperature boundaries in two-dimensional simulations [35], and in three-dimensional simulations both with rotation [36] and without [37], albeit at significantly lower Rayleigh numbers. Recent direct numerical simulations at much higher Rayleigh numbers (and  $Pr = .7$ ) reported  $Nu \sim Ra^{1/3}$  over nearly four decades up to  $Ra = 10^{14}$  in a three-dimensional cylindrical cell of aspect ratio 1/2 [38]. It remains to be seen whether the heat transport in two-dimensional Rayleigh-Bénard convection ever deviates from the  $Nu \sim Ra^{2/7}$  scaling at higher Rayleigh numbers.

In summary, the high-resolution simulation results reported here suggest that the plate conductivity plays no

significant role in the  $Nu \sim Ra^{2/7}$  transport law at asymptotically high Ra in two dimensions with periodic side conditions at  $Pr = 1$ . However, it remains to be determined how various combinations of geometry, side-wall conditions, plate conductivity and the spatial dimension affect the bulk transport in high-resolution simulations at asymptotically high Rayleigh numbers.

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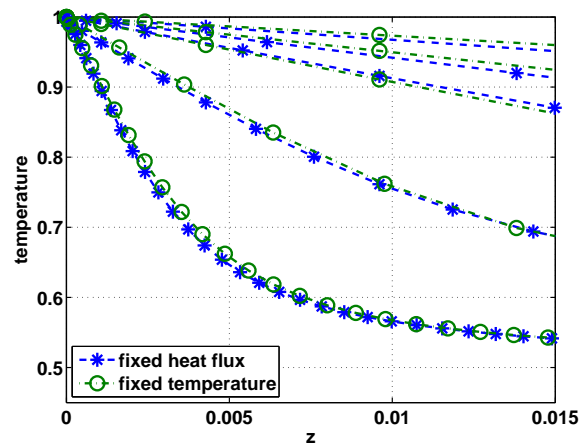


FIG. 5: Time averaged temperature profiles (from right to left):  $Ra = 1.03 \times 10^4$ ,  $1.03 \times 10^5$ ,  $1.21 \times 10^6$ ,  $1.22 \times 10^8$ , and  $1.05 \times 10^{10}$ . The discrete data indicate the spatial discretization used to resolve the boundary layers.

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[www.math.umass.edu/~johnston/RayBen.html](http://www.math.umass.edu/~johnston/RayBen.html)

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